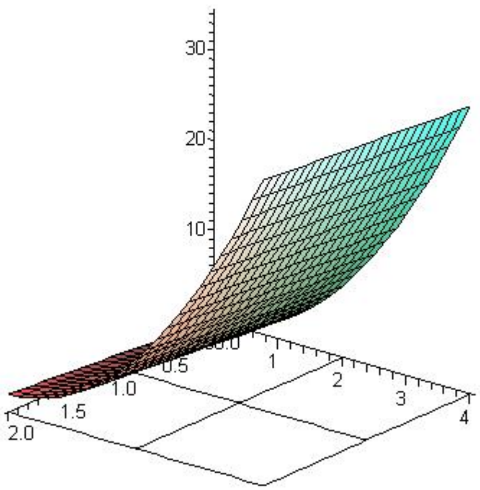
Close today: 14.7
Close Tue: 15.1, 15.2
Close Thur: 15.3
Office Hours Today: 12:30-2:30pm (MSC)

15.1-15.2 Intro to double Integrals

Goal: Give a definition for the volume between a *given surface* and a *given region* on the *xy*-plane.

In all of ch. 15, you are given two things:

1. A surface: z = f(x,y)
2. A region drawn on the *xy*-plane.



Example:
The volume under

$$z = f(x,y) = x + 2y^2$$

and above
R = [0,2] x [0,4] = {(x,y) : 0 ≤ x ≤ 2, 0 ≤ y ≤ 4 }

(a) Break the region R into m = 2 columns and n = 2 rows; 4 sub-regions;

(a) Approx. using a rectangular box over each region (use *upper-right* endpts).

Formally, we define:

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

= the `signed' volume between *f(x,y)* and the *xy*-plane over *R*.

If *f(x,y)* is above the *xy*-plane it is positive.
If *f(x,y)* is below the *xy*-plane it is negative.

General Notes and Observations:

z = *f(x,y)* = height on surface
R = the region on the *xy*-plane

ΔA = area of base = $\Delta x \Delta y = \Delta y \Delta x$

$f(x_{ij}, y_{ij})\Delta A$ = (height)(area of base)
= volume of one approximating box

Units of $\iint_R f(x,y)dA$ are
(units of f(x,y))(units of x)(units of y)

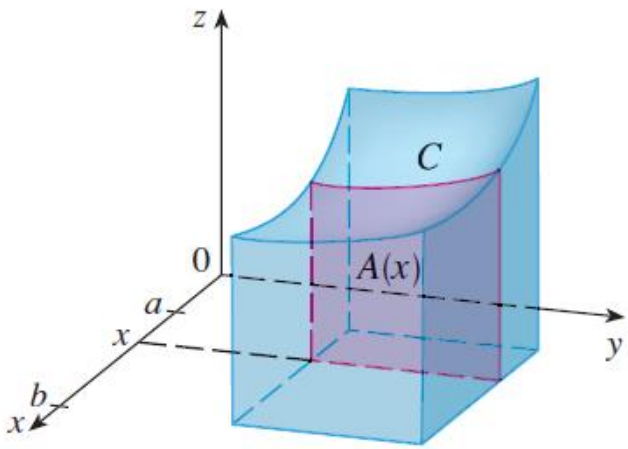
Quick application note:

$$\iint_R 1dA = \text{"Area of R"}\text{, and}$$

Iterated Integrals

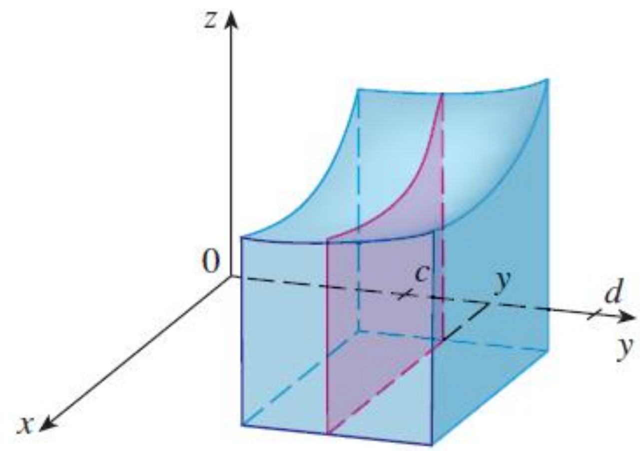
If you fix x: The area under this curve is

$$\int_c^d f(x,y)\,dy = \text{"cross sectional area under the surface at this fixed } x \text{ value"}$$



If you fix y: The area under this curve

$$\int_a^b f(x,y)\,dx = \text{"cross sectional area under the surface at this fixed } y \text{ value"}$$



From Math 125,

$$\text{Vol} = \int_a^b \text{Area(x)}\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$\text{Vol} = \int_c^d \text{Area}(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Quick Example: Evaluate

$$(a)\ \int_{2}^{6}\left(\int_{1}^{8} y\,dx\right)dy$$

$$(b)\ \int_{2}^{6}\left(\int_{1}^{8} 1\,dx\right)dy$$

Examples (like 15.2 HW):

1. Find the volume under
   $z = x + 2y^2$ and
   above $0 \leq x \leq 2, \quad 0 \leq y \leq 4$

2. $\int_{0}^{3}\int_{0}^{1} 2xy\sqrt{x^2+y^2}\,dxdy$

3. Find the double integral of

$$f(x,y) = y\cos(x+y)$$

over the rectangular region

$$0 \le x \le \pi, \quad 0 \le y \le \pi/2$$

15.2 Double Integrals over General Regions

For the rectangular region, *R*, given by

$$a \le x \le b \ , \qquad c \le y \le d$$

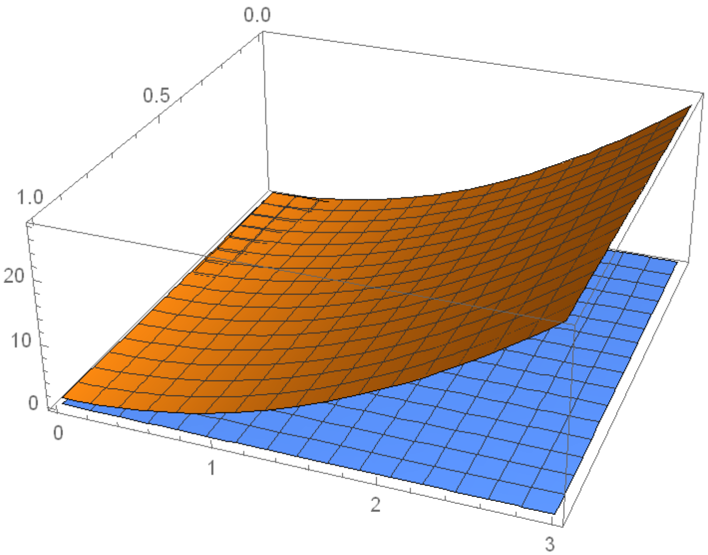
we learned:

$$\iint_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$
$$= \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

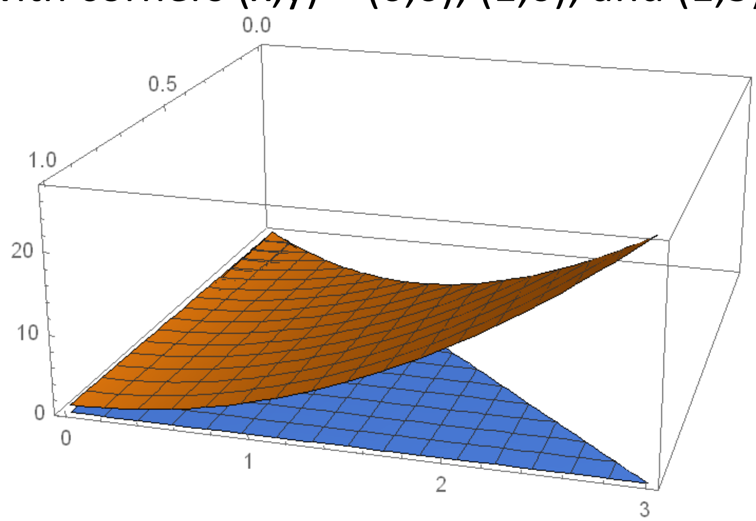
In 15.2, we discuss regions, *R*, other than rectangles.

Type 1 Regions (Top/Bot)	Type 2 Regions (Left/Right)
Given *x* in the range, $a \le x \le b$, we have $g_1(x) \le y \le g_2(x)$	Given *y* in the range, $c \le y \le d$, we have $h_1(y) \le x \le h_2(y)$
$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$	$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$

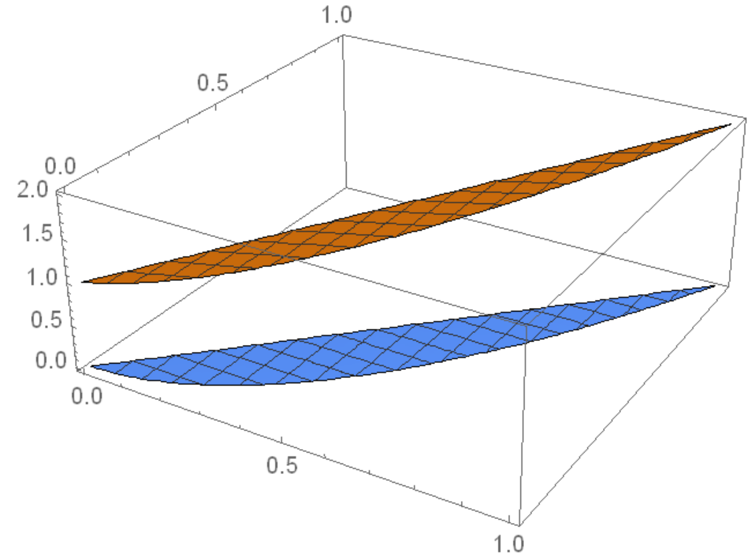
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$



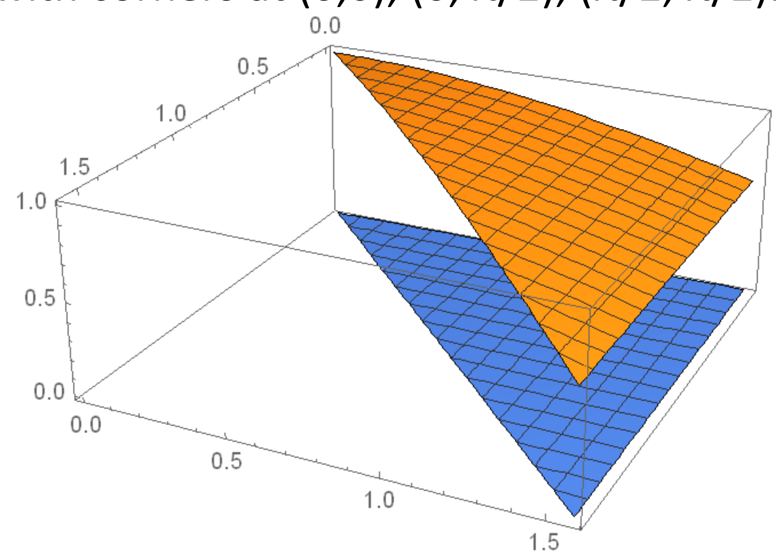
The surface $z = x + 3y^2$ over the triangular region with corners $(x,y) = (0,0)$, $(1,0)$, and $(1,3)$.



The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.



The surface $z = \sin(y)/y$ over the triangular region with corners at $(0,0)$, $(0, \pi/2)$, $(\pi/2, \pi/2)$.



Examples:

1. Let D be the triangular region in the xy-plane with corners (0,0), (1,0), (1,3).

$$\text{Evaluate} \iint_D x + 3y^2 dA$$

2. Find the volume of the solid bounded by the surfaces $z = x + 1, y = x^2$, $y = 2x, z = 0$.

3. Draw the region of integration for

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin(y)}{y}\, dy\, dx$$

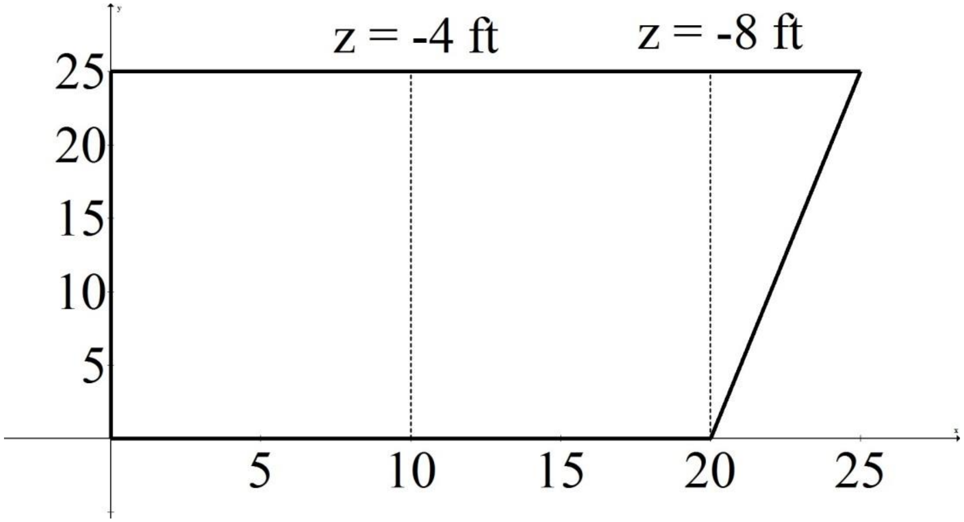
then switch the order of integration.

4. Switch the order of integration for

$$\int_{0}^{4} \int_{\sqrt{x}}^{2} \sin(y^3)\, dy dx$$

An applied problem:

Your swimming pool has the following shape (viewed from above)



The bottom of the pool is a plane with depths as indicated (the pool gets deeper in a linear way from left-to-right)

Solution:

1. Describe the surface (what is *z*?):
   Slope in y-direction = 0
   Slope in x-direction = -4/10 = -0.4
   Also the plane goes through (0, 0, 0)
   Thus, the plane that describes the bottom of the pool is: **z = -0.4x + 0y**

2. Describe the region in *xy*-plane:
   The line on the right goes through (20,0) and (25,25), so it has slope = 5 and it is given by the equation
   y = 5(x-20) = 5x – 100
   or **x = (y+100)/5 = 1/5 y + 20**
   The best way to describe this region is by thinking of it as a left-right region.
   On the left, we always have x = 0
   On the right, we always have x = 1/5 y + 20

Therefore, we have

$$\int_0^{25} \left(\int_0^{\frac{1}{5}y+20} -0.4\,x\,dx \right) dy = -741.\overline{6}\ \ \text{ft}^3$$